Ethiopian Journal of Indigenous Knowledge and Applied Sciences

Eth. J. Indig. Know. Appl. Sci.
Vol. 3, No. 1, 2025, 16-20

Skew Almost Distributive Lattices

Yeshiwas Mebrate[1]* , Berhanu Assaye Alaba[2] , Mihret Alamneh Taye[3]

[123]Department of Mathematics, College of Science, Bahir Dar University, Ethiopia

Abstract

Abstract: In this paper, we introduce the concept of Skew Almost Distributive Lattices (skew ADLs). We define a relation $\boldsymbol{\Theta}$ on a skew ADL $\boldsymbol{S}$ so that each congruence class is the maximal rectangular subalgebra and $\boldsymbol{S}/\boldsymbol{\Theta}$ is the maximal lattice image of $\boldsymbol{S}$. Further, the calculative properties of a skew ADL are studied.

Keywords: Lattice; Almost distributive lattice; Skew lattice; Skew almost distributive lattice.

1. Introduction

The concept of an ADL was introduced in 1981 by Swammy, U. M. and Rao, G. C. (7) as a common abstraction of almost the existing ring theoretic generalizations of a Boolean algebras. An ADL is an algebra $(L,\vee,\wedge,0)$ of type $(2,0,0)$ which satisfies all the axioms of a distributive lattice with zero except possibly commutativity of $\vee$ or commutativity of $\wedge$ or right distributivity of $\vee$ over $\wedge$ or the absorption law $(\sigma \wedge \delta) \vee \sigma = \sigma$. It was observed that, the set of all principal ideals of an ADL becomes a distributive lattice through which one can extend many concepts existing in the class of distributive lattices to the class of ADLs.

In 1989 Jonathan Leech laid the foundation of modern theory of skew lattices (4). He defines a skew lattice is an algebra $(L,\vee,\wedge)$ in which both $\vee$ and $\wedge$ satisfies the idempotent laws, associative laws and absorption laws. For a skew lattice $(L,\vee,\wedge)$ and for $\delta,\sigma \in L$, δ and σ are said to be equivalent, denoted $\delta \equiv \sigma$, whenever $\delta \vee \sigma \vee \delta = \delta$ and $\sigma \vee \delta \vee \sigma = \sigma$.

Motivated by the above results we studied about skew lattices in the class of ADLs and we define a skew ADL. We give different results on which a skew ADL is a rectangular lattice. Moreover, for a skew ADL S, we showed that S/Θ is a maximal lattice image of S where Θ is a congruence relation.

2. Preliminary

In this section, we give results that we use in the sequel.

Definition 2.1 [1, 2] An algebra $(L,\vee,\wedge)$ of type $(2,2)$ is called a lattice if it satisfies the following identities for all $\sigma,\delta,\omega \in L$:

[1] Idempotency: $\sigma \wedge \sigma = \sigma$ and $\sigma \vee \sigma = \sigma$

[2] Commutativity: $\sigma \wedge \delta = \delta \wedge \sigma$ and $\sigma \vee \delta = \delta \vee \sigma$

[3] Associativity: $(\sigma \wedge \delta) \wedge \omega = \sigma \wedge (\delta \wedge \omega)$ and $(\sigma \vee \delta) \vee \omega = \sigma \vee (\delta \vee \omega)$

[4] Absorption laws: $\sigma \wedge (\sigma \vee \delta) = \sigma$ and $\sigma \vee (\sigma \wedge \delta) = \sigma$.

In any lattice $(L,\vee,\wedge)$, the following identities are equivalent [6]:

[1] $\sigma \wedge (\delta \vee \omega) = (\sigma \wedge \delta) \vee (\sigma \wedge \omega)$

[2] $(\sigma \vee \delta) \wedge \omega = (\sigma \wedge \omega) \vee (\delta \wedge \omega)$

[3] $\sigma \vee (\delta \wedge \omega) = (\sigma \vee \delta) \wedge (\sigma \vee \omega)$

[4] $(\sigma \wedge \delta) \vee \omega = (\sigma \vee \omega) \wedge (\delta \wedge \omega)$, for all $\sigma,\delta,\omega \in L$.

Definition 2.2 [2] A lattice $(L,\vee,\wedge)$ satisfying any one of the above four identities is called a distributive lattice.

Definition 2.3 [3] Let X be a nonempty set and Θ be a binary relation on X (that is $\Theta \subseteq X \times X$). Then Θ is said to be an equivalence relation on X if Θ satisfies the following conditions:

[1] Reflexive: $(\delta,\delta) \in \Theta$ for all $\delta \in X$

[2] Symmetric: $(\delta,\sigma) \in \Theta$ implies that $(\sigma,\delta) \in \Theta$ for all $\delta,\sigma \in X$

[3] Transitive: $(\delta,\sigma) \in \Theta$ and $(\sigma,\omega) \in \Theta$ imply that $(\delta,\omega) \in \Theta$ for all $\delta,\sigma,\omega \in X$.

For $(\delta,\sigma) \in \Theta$, We write $\delta\ \Theta\ \sigma$.

*Corresponding author: yeshiwasmebrat@gmail.com
Received: Jun. 20/2024, Revised **Nev. 10/2024, Accepted**: Dec. 26/2024, **Published**: Feb 1/2024.

Definition 2.4 [5] An algebra $(L,\vee,\wedge,0)$ of type (2,2,0) is called an ADL with 0 if it satisfies the following axioms:

[1] $\delta \vee 0 = \delta$
[2] $0 \wedge \delta = 0$
[3] $(\delta \vee \sigma) \wedge z = (\delta \wedge \omega) \vee (\sigma \wedge \omega)$
[4] $\delta \wedge (\sigma \vee \omega) = (\delta \wedge \sigma) \vee (\delta \wedge \omega)$
[5] $\delta \vee (\sigma \wedge \omega) = (\delta \vee \sigma) \wedge (\delta \vee \omega)$
[6] $(\delta \vee \sigma) \wedge \sigma = \sigma$
for all $\delta, \sigma, \omega \in L$.

It can be seen directly that every distributive lattice is an ADL. Here onwards by L we mean an ADL $(L,\vee,\wedge,0)$. For any $\sigma, \delta \in L$, we say that σ is less than or equal to δ and write $\sigma \leq \delta$, if $\sigma \wedge \delta = \sigma$. Then "$\leq$" is a partial ordering on L. The following hold in any ADL L.

Theorem 2.5 [5] Let L be an ADL with 0. Then for any $\omega, \delta, \sigma, z \in L$, the following conditions hold:

[1] $\delta \vee \sigma = \delta \Leftrightarrow \delta \wedge \sigma = \sigma$
[2] $\delta \vee \sigma = \sigma \Leftrightarrow \delta \wedge \sigma = \delta$
[3] $\delta \wedge \sigma = \sigma \wedge \delta = \delta$ whenever $\delta \leq \sigma$
[4] $\wedge$ is associative
[5] $\delta \wedge \sigma \wedge z = \sigma \wedge \delta \wedge z$
[6] $(\delta \vee \sigma) \wedge z = (\sigma \vee \delta) \wedge z$
[7] $\delta \wedge \sigma \leq \sigma$ and $\delta \leq \delta \vee \sigma$
[8] $\delta \wedge (\sigma \wedge \delta) = \sigma \wedge \delta$ and $\delta \vee (\delta \vee \sigma) = \delta \vee \sigma = (\delta \vee \sigma) \vee \sigma$
[9] $\delta \wedge \delta = \delta$ and $\delta \vee \delta = \delta$
[10] $\delta \wedge 0 = 0$ and $0 \vee \delta = \delta$
[11] $(\omega \vee (\delta \vee \sigma)) \wedge z = ((\omega \vee \delta) \vee \sigma) \wedge z$
[12] If $\delta \leq \omega$ and $\sigma \leq \omega$, then $\delta \wedge \sigma = \sigma \wedge \delta$ and $\delta \vee \sigma = \sigma \vee \delta$.

It can be observed that an ADL L satisfies all the properties of a distributive lattice except possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(\sigma \wedge \delta) \vee \sigma = \sigma$. Any one of these properties convert L in to a distributive lattice.

Theorem 2.6 [5] Let $(L,\vee,\wedge,0)$ be an ADL with 0. Then the following are equivalent.

[1] $(L,\vee,\wedge,0)$ is a distributive lattice
[2] $\sigma \vee \delta = \delta \vee \sigma$ for all $\sigma, \delta \in L$
[3] $\sigma \wedge \delta = \delta \wedge \sigma$ for all $\sigma, \delta \in L$
[4] $(\sigma \wedge \delta) \vee \omega = (\sigma \vee \omega) \wedge (\delta \vee \omega)$, for all $\omega, \sigma, \delta \in L$.

Theorem 2.7: [7] Let $(L,\vee,\wedge,0)$ be an ADL. Then for any $\sigma, \delta, \omega \in L$ with $\sigma \leq \delta$, we have the following conditions:

[1] $\sigma \wedge \omega \leq \delta \wedge \omega$
[2] $\omega \wedge \sigma \leq \omega \wedge \delta$
[3] $\omega \vee \sigma \leq \omega \vee \delta$.

Theorem 2.8: [5] Let L be an ADL. For any $m \in L$ the following are equivalent.

[1] m is maximal element
[2] $m \vee \delta = m$ for all $\delta \in L$
[3] $m \wedge \delta = \delta$ for all $\delta \in L$.

For any $\omega, \sigma \in L$ and a partial ordering $\leq$ on L, the set $\{\delta \in L | \sigma \leq \delta \leq \omega\}$ is an interval denoted by $[\sigma, \omega]$.

Theorem 2.9: [5] Let L be an ADL. Then the following are equivalent.

[1] L is associative
[2] $\Theta_\sigma = \{(\delta, \sigma) \in R \times R : \sigma \vee \delta = \sigma \vee \sigma\}$ is a congruence relation for all $\sigma \in L$
[3] L is a subdirect product of ADLs in each of which there are at most two nonzero elements and every nonzero element is maximal.

Note: In an ADL $(L,\vee,\wedge,0)$ the associative property with respect to $\wedge$ holds, that is for any $\sigma, \delta, \omega \in L$, $\sigma \wedge (\delta \wedge \omega) = (\sigma \wedge \delta) \wedge \omega$. Whereas the associative property with respect to $\vee$, that is, for any $\sigma, \delta, \omega \in L$, $\sigma \vee (\delta \vee \omega) = (\sigma \vee \delta) \vee \omega$ is not known so far.

Theorem 2.10: [7] In an ADL $(L,\vee,\wedge,0)$, the following are equivalent.

[1] L is a distributive lattice
[2] $(L, \leq)$ is a directed above poset
[3] $\wedge$ is commutative
[4] $\vee$ is commutative
[5] $\vee$ is right distributive over $\wedge$
[6] $\Theta = \{(\sigma, \delta) \in L : \delta \wedge \sigma = \sigma\}$ is antisymmetric.

Definition 2.11 [4]: A skew lattice is an algebra $(L,\vee,\wedge)$ of type (2,2) such that $\wedge$ and $\vee$ are both associative and satisfy the following absorption laws: $\delta \wedge (\delta \vee \sigma) = \delta = \delta \vee (\delta \wedge \sigma)$ and $(\delta \wedge \sigma) \vee \sigma = \sigma = (\delta \vee \sigma) \wedge \sigma$ for all $\delta, \sigma \in L$.

Definition 2.12 [4]: A skew lattice is called strongly distributive if for all $\delta, \sigma, \omega \in L$ it satisfies the following identities: $\delta \wedge (\sigma \vee \omega) = (\delta \wedge \sigma) \vee (\delta \wedge \omega)$ and $(\delta \vee \sigma) \wedge \omega = (\delta \wedge \omega) \vee (\sigma \wedge \omega)$; and it is called co-strongly distributive if it satisfies the identities: $\delta \vee (\sigma \wedge \omega) = (\delta \vee \sigma) \wedge (\delta \vee \omega)$ and $(\delta \wedge \sigma) \vee \omega = (\delta \vee \omega) \wedge (\sigma \vee \omega)$.

Definition 2.13 [4]: A skew Lattice L is called normal if $\omega \wedge \delta \wedge \sigma \wedge z = \omega \wedge \sigma \wedge \delta \wedge z$ and it is called co-normal if $\omega \vee \delta \vee \sigma \vee z = \omega \vee \sigma \vee \delta \vee z$ for all $\omega, \delta, \sigma, z \in L$.

3. Skew Almost Distributive Lattices

In this section we introduce different properties of skew almost distributive lattices.

Definition 3.1 Let $(L,\vee,\wedge)$ be an ADL and S is a nonempty subset of L. An algebra $(S,\vee,\wedge)$ is called a Skew Almost Distributive Lattice (skew ADL) if $\delta \vee (\sigma \vee \omega) = (\delta \vee \sigma) \vee \omega$ for all $\delta, \sigma, \omega \in S$.
From the property of ADL, a skew ADL is strongly distributive skew lattice. We call a skew ADL S is co-strongly distributive skew ADL if $(\delta \wedge \sigma) \vee \omega = (\delta \vee \omega) \wedge (\sigma \vee \omega)$ for all $\delta, \sigma, \omega \in S$ and one can simply observe that a co-strongly distributive skew ADL with zero is a distributive lattice. Skew ADLs are skew lattices and hence we can apply all concepts of skew lattices to skew ADLs, for if a skew ADL is biregular.

Definition 3.2 A rectangular skew ADL is a skew ADL S such that $\delta \wedge \sigma = \sigma \vee \delta$ for all non zero elements $\delta, \sigma \in S$.
Lemma 3.3 Let S be a skew ADL. Then, S is normal.
Proof: Suppose S be a skew ADL. Let $\omega, \delta, \sigma, z \in S$. Then, from Theorem 2.5 (5) we obtain that $\omega \wedge \delta \wedge \sigma \wedge z = \omega \wedge \sigma \wedge \delta \wedge z$. This shows that S is normal.

Lemma 3.4 A rectangular skew ADL is co-normal.

Proof: Let S be a rectangular skew ADL and $\omega, \delta, \sigma, z \in S$. Then $\delta \vee \sigma \vee z \vee \omega = \omega \wedge z \wedge \sigma \wedge \delta = \omega \wedge \sigma \wedge z \wedge \delta = \delta \vee z \vee \sigma \vee \omega$.
Hence, S is co-normal.

Lemma 3.5 A rectangular skew ADL with 0 is a discrete ADL.
Proof: Let S be a rectangular skew ADL and $\delta, \sigma \in S$. Then, by definition we obtain that $\delta \wedge \sigma = \sigma \vee \delta$ for all $\delta, \sigma \in S$ such that $\delta \neq 0$ and $\sigma \neq 0$. Since $\sigma \wedge \delta = \delta \wedge \sigma \wedge \delta = (\sigma \vee \delta) \wedge \delta = \delta$, using the absorption law it follows that $\sigma \vee \delta = \sigma \vee (\sigma \wedge \delta) = \sigma$. Which shows that S is a discrete ADL.
Theorem 3.6 Let S be a skew ADL with 0. Then S is a rectangular skew ADL if and only if $\delta \vee \sigma = \delta$ for all $\delta, \sigma \in S$.
Proof: Suppose S be a rectangular skew ADL. It follows that $\sigma \vee \delta = \delta \wedge \sigma$ for all $\delta, \sigma \in S$ such that $\delta \neq 0$ and $\sigma \neq 0$. From Lemma 3.5, S is discrete ADL. Hence, $\delta \vee \sigma = \delta$ for all nonzero $\delta, \sigma \in S$. Conversely, suppose $\delta \vee \sigma = \delta$. Then, it follows that $\delta \vee \sigma \vee \delta = \delta$. Hence, $\delta \vee \sigma = ((\sigma \wedge \delta) \vee \delta) \vee (\sigma \vee (\sigma \wedge \delta)) = (\sigma \wedge \delta) \vee (\delta \vee \sigma) \vee (\sigma \wedge \delta) = \sigma \wedge \delta$. Therefore, S is rectangular skew ADL.
Theorem 3.7 Let S be a skew ADL. A relation Θ defined on S by

$$\Theta = \{(\delta, \sigma) \in S \times S \mid \delta \wedge \sigma = \sigma \text{ and } \sigma \wedge \delta = \delta\}$$

is a congruence relation.
Proof: Let $\delta, \sigma, \omega \in S$. Then,
[1] $\delta \wedge \delta = \delta$. Hence, Θ is reflexive.
[2] Let $\delta \,\Theta\, \sigma$. Then $\delta \wedge \sigma = \sigma$ and $\sigma \wedge \delta = \delta$. Hence, $\sigma \,\Theta\, \delta$.
[3] Let $\delta \,\Theta\, \sigma$ and $\sigma \,\Theta\, \omega$. Then, we have $\delta \wedge \sigma = \sigma, \sigma \wedge \delta = \delta, \sigma \wedge \omega = \omega$ and $\omega \wedge \sigma = \sigma$. Then,

$$\delta \wedge \omega = \delta \wedge (\sigma \wedge \omega) = (\delta \wedge \sigma) \wedge \omega = \sigma \wedge \omega = \omega$$

and

$$\omega \wedge \delta = \omega \wedge (\sigma \wedge \delta) = (\omega \wedge \sigma) \wedge \delta = \sigma \wedge \delta = \delta.$$

Hence, $\delta \,\Theta\, \omega$. Thus, Θ is an equivalence relation.
[4] Assume that $\delta \,\Theta\, \sigma$ and $\omega \,\Theta\, z$. Then,

$$\begin{aligned}(\delta \vee w) \wedge (\sigma \vee z) &= ((\delta \vee \omega) \wedge \sigma) \vee ((\delta \vee \omega) \wedge z) \\ &= ((\delta \wedge \sigma) \vee (\omega \wedge \sigma)) \vee ((\delta \wedge z) \vee (\omega \wedge z)) \\ &= (\sigma \vee (\omega \wedge \sigma)) \vee ((\delta \wedge z) \vee z) \\ &= ((\sigma \vee \omega) \wedge \sigma) \vee z \\ &= \sigma \vee z.\end{aligned}$$

Similarly, $(\sigma \vee z) \wedge (\delta \vee \omega) = \delta \vee \omega$. Hence, we have $(\delta \vee \omega \,\Theta\, (\sigma \vee z)$. Also,

$$\begin{aligned}(\delta \wedge \omega) \wedge (\sigma \wedge z) &= \delta \wedge \sigma \wedge \omega \wedge z = \sigma \wedge \omega \wedge z \\ &= \sigma \wedge z\end{aligned}$$

and

$$\begin{aligned}(\sigma \wedge z) \wedge (\delta \wedge \omega) &= z \wedge \sigma \wedge \delta \wedge \omega = z \wedge \delta \wedge \omega \\ &= \delta \wedge z \wedge \omega = \delta \wedge \omega.\end{aligned}$$

Hence, we have $(\delta \wedge \omega) \,\Theta (\sigma \wedge z)$.
Therefore, Θ is a congruence relation on S.

Lemma 3.8 Let S be a skew ADL and $\delta, \sigma \in S$. For Θ given by Theorem 3.7, $\delta \,\Theta\, \sigma$ if and only if $\delta \vee \sigma = \sigma \wedge \delta$.
Proof: Let $\delta \,\Theta\, \sigma$. Then,

$$\begin{aligned}\delta \vee \sigma &= (\sigma \vee \delta \vee \sigma) \wedge (\delta \vee \sigma) \\ &= (\sigma \vee (\sigma \wedge \delta)) \wedge (\delta \vee \sigma) \\ &= \sigma \wedge (\delta \vee \sigma).\end{aligned}$$

This implies that

$$\begin{aligned}\sigma \wedge \delta &= \sigma \wedge \delta \wedge (\delta \vee \sigma) \\ &= \sigma \wedge \delta \wedge \big(\sigma \wedge (\delta \vee \sigma)\big) \\ &= \sigma \wedge (\delta \vee \sigma) \\ &= \delta \vee \sigma.\end{aligned}$$

Conversely if $\delta \vee \sigma = \sigma \wedge \delta$, then $\sigma \wedge \delta = \delta \wedge \sigma \wedge \delta = \delta \wedge (\delta \vee \sigma) = \delta$ and $\delta \wedge \sigma = \sigma \wedge \delta \wedge \sigma = \sigma \wedge (\sigma \vee \delta) = \sigma$ which implies that $\delta \,\Theta\, \sigma$.

Theorem 3.9 Let S be a rectangular skew ADL and $\delta, \sigma, \omega \in S$. Then the following conditions hold:
[1] $\delta \wedge \omega = \sigma \wedge \omega$
[2] $(\delta \vee \omega) \wedge \sigma = \sigma$ and $\sigma \vee (\delta \wedge \omega) = \sigma$
[3] S is a lattice if and only if it is a singleton.
Proof: Suppose S be a rectangular skew ADL and $\delta, \sigma, \omega \in S$. Then,
[1] Using regularity we get

$$\begin{aligned}\delta \wedge \omega &= ((\sigma \vee \delta) \wedge \delta) \wedge ((\sigma \vee \omega) \wedge \omega) \\ &= (\delta \wedge \sigma \wedge \delta) \wedge (\omega \wedge \sigma \wedge \omega)\end{aligned}$$

$= \delta \wedge \sigma \wedge \delta \wedge \omega \wedge \sigma \wedge \omega$
$= \delta \wedge \sigma \wedge \delta \wedge \sigma \wedge \omega \wedge \sigma \wedge \omega$
$= \delta \wedge (\sigma \wedge \delta \wedge \sigma) \wedge (\omega \wedge \sigma \wedge \omega)$

$= \delta \wedge \sigma \wedge \omega.$

By symmetry we also have $\sigma \wedge \omega = \sigma \wedge \delta \wedge \omega$.
Since $\delta \wedge \sigma \wedge \omega = \sigma \wedge \delta \wedge \omega$ we conclude that $\delta \wedge \omega = \sigma \wedge \omega$.

[2] Using Theorem 3.6 we obtain that

$$\delta \vee \sigma \vee \omega = (\delta \vee \sigma) \vee \omega = \delta \vee \omega$$

and hence

$$(\delta \vee \omega) \wedge \sigma = (\delta \vee \sigma \vee \omega) \wedge \sigma = (\delta \vee \omega \vee \sigma) \wedge \sigma = \sigma.$$

Since $\delta \wedge \omega = \omega \vee \delta$, we have

3. $\sigma \vee (\delta \wedge \omega) = (\delta \wedge \omega) \wedge \sigma = (\omega \vee \delta) \wedge \sigma = (\delta \vee \omega) \wedge \sigma = \sigma$
4. [3] Suppose S be a lattice. Then, by Theorem 3.6 we obtain that $\delta = \delta \vee \sigma = \sigma \vee \delta = \sigma$ for all $\delta, \sigma \in S$. Therefore, S is a singleton. The converse is direct.

A skew ADL is said to be left cancellative if it satisfies the implication $\delta \vee \sigma = \delta \vee \omega$ and $\delta \wedge \sigma = \delta \wedge \omega$ imply $\sigma = \omega$. It is called right cancellative if it satisfies $\delta \vee \omega = \sigma \vee \omega$ and $\delta \wedge \omega = \sigma \wedge \omega$ imply $\delta = \sigma$. If a skew ADL is both right and left cancellative then it is said to be cancellative.

Theorem 3.10 Let S be a skew ADL. Then S is right cancellative skew ADL.

Proof: Suppose S be a skew ADL and $\delta, \sigma, \omega \in S$ such that $\delta \wedge \omega = \sigma \wedge \omega$ and $\delta \vee \omega = \sigma \vee \omega$. Then,

$$\begin{aligned} \delta &= \delta \vee (\delta \wedge \omega) \\ &= \delta \vee (\sigma \wedge \omega) \\ &= (\delta \vee \sigma) \wedge (\delta \vee \omega) \\ &= (\delta \vee \sigma) \wedge (\sigma \vee \omega) \\ &= (\sigma \vee \delta) \wedge (\sigma \vee \omega) \\ &= \sigma \vee (\delta \wedge \omega) \\ &= \sigma \vee (\sigma \wedge \omega) \\ &= \sigma. \end{aligned}$$

Hence, S is right cancellative skew ADL

Corollary 3.11 Let S be a rectangular skew ADL. Then, S is cancellative.

Proof: Suppose S be a rectangular skew ADL such that $\omega \vee \delta = \omega \vee \sigma$ and $\omega \wedge \delta = \omega \wedge \sigma$ for all $\delta, \sigma, \omega \in S$. Then, we have

$$\begin{aligned} \delta &= (\omega \vee \delta) \wedge \delta \\ &= (\delta \wedge \omega) \wedge \delta \\ &= \delta \wedge (\omega \wedge \delta) \\ &= \delta \wedge (\omega \wedge \sigma) \\ &= (\delta \wedge \omega) \wedge \sigma \\ &= (\omega \vee \delta) \wedge \sigma \\ &= (\omega \vee \sigma) \wedge \sigma \\ &= \sigma. \end{aligned}$$

This shows that S is left cancelletive and hence it is cancellative.

Lemma 3.12 Let S be a skew ADL and Θ be a congruence relation on S given by Theorem 3.7. Then the congruence class $[\delta]\Theta$ of δ is rectangular skew ADL.

Proof: Suppose S be a skew ADL and $\delta, \sigma, \omega \in S$ such that $\sigma, \omega \in [\delta]\Theta$. Since $\sigma \,\Theta\, \omega$ we obtain that $\sigma \vee \omega = \sigma = \omega \wedge \sigma$ and hence $[\delta]\Theta$ is a rectangular skew ADL.

Theorem 3.13 Let S be a skew ADL and Θ be a relation given by Theorem 3.7. Then, S/Θ is a maximal lattice image of S.

Proof: Suppose S be a skew ADL. From Theorem 3.7, Θ is a congruence relation on S. Consider S/Θ and for $\delta, \sigma \in S$ let $[\delta]\Theta, [\sigma]\Theta \in S/\Theta$. Define $\wedge$ and $\vee$ on S/Θ by $[\delta]\Theta \wedge [\sigma]\Theta = [\delta \wedge \sigma]\Theta$ and $[\delta]\Theta \vee [\sigma]\Theta = [\delta \vee \sigma]\Theta$. Then, for any $\omega \in S$,

$\omega \in [\delta \vee \sigma]\Theta \iff (\delta \vee \sigma) \wedge \omega = \omega$ and $\omega \wedge (\delta \vee \sigma) = \delta \vee \sigma$
$\iff (\sigma \vee \delta) \wedge \omega = \omega$ and $\omega \wedge (\sigma \vee \delta) = \sigma \vee \delta$

$\iff \omega \in [\sigma \vee \delta]\Theta$

Hence, $[\delta \vee \sigma]\Theta = [\sigma \vee \delta]\Theta$. Thus, $[\delta]\Theta \vee [\sigma]\Theta = [\sigma]\Theta \vee [\delta]\Theta$.

Similarly, $\omega \in [\delta \wedge \sigma]\Theta \iff \delta \wedge \sigma \wedge \omega = \omega$ and $\omega \wedge (\delta \wedge \sigma) = \delta \wedge \sigma$

$\iff \sigma \wedge \delta \wedge \omega = \omega$ and $\omega \wedge (\sigma \wedge \delta) = \sigma \wedge \delta$
$\iff \omega \in [\sigma \wedge \delta]\Theta.$

Hence, $[\delta \wedge \sigma]\Theta = [\sigma \wedge \delta]\Theta$. Thus, $[\delta]\,\Theta \wedge [\sigma]\,\Theta = [\sigma]\,\Theta \wedge [\delta]\,\Theta$. Which shows that S/Θ is commutative and therefore it is a lattice. Now, consider a congruence relation β on S such that S/β is a lattice. Suppose $\delta \,\Theta\, \sigma$. Clearly,

$\delta \wedge \sigma = \sigma \; \sigma nd \; \sigma \wedge \delta = \delta \implies [\delta \wedge \sigma]\,\beta = [\sigma]\,\beta$ and $[\sigma \wedge \delta]\,\beta = [\delta]\,\beta$
$\implies [\delta]\,\beta \wedge [\sigma]\,\beta = [\sigma]\,\beta$ and $[\sigma]\,\beta \wedge [\delta]\,\beta = [\delta]\,\beta$
$\implies [\delta]\,\beta = [\sigma]\,\beta$, since \$/ β is a lattice

$\implies \delta \,\beta\, \sigma.$

Therefore, $\Theta \subseteq \beta$.

Suppose H be a lattice image of S. Then there exist an epimorphism $f: S \longrightarrow H$. Define a relation α on S by $\alpha = \{(\delta, \sigma) \in S^2 | f(\delta) = f(\sigma)\}$. α is reflexive, symmetric and transitive. Let $(\delta_1, \sigma_1), (\delta_2, \sigma_2) \in \alpha$. Since f is homomorphism we have

$$\begin{aligned} f(\delta_1 \vee \delta_2) &= f(\delta_1) \vee f(\delta_2) \\ &= f(\sigma_1) \vee f(\sigma_2) \\ &= f(\sigma_1 \vee \sigma_2), \end{aligned}$$

which shows that $(\delta_1 \vee \delta_2)\ \alpha\ (\sigma_1 \vee \sigma_2)$ similarly $(\delta_1 \wedge \delta_2)\ \alpha\ (\sigma_1 \wedge \sigma_2)$\$. Hence α is a congruence relation on S. Since a homomorphic image of S is given by S/ϕ where ϕ is a congruence relation on S [5],S$/\alpha$ is the lattice image of S. Consider the set $\mathcal{L}$ of lattices given by $\mathcal{L} = \{S/\phi \mid \phi$ is a congruence on S and $\Theta \subseteq \phi\}$. Now, take congruence relations Θ and β discussed above and assume that $S/\Theta \subseteq S/\beta$. Let $\sigma, \delta \in S$ such that $\sigma\beta\delta$. Since $\Theta \subseteq \beta$ and $S/\Theta \subseteq S/\beta$ there exist $\sigma', \delta' \in S$ such that $\sigma \in [\sigma]\Theta = [\sigma']\beta$ and $\delta \in [\delta]\Theta = [\delta']\beta$. Thus, $\sigma\Theta\sigma'$, $\sigma\beta\sigma'$ and $\delta\Theta\delta'$, $\delta\beta\delta'$. Hence,

$$\sigma\beta\delta, \delta\beta\delta' \Longrightarrow \sigma\beta\delta'$$

$$\Longrightarrow \sigma \in [\delta']\beta = [\delta]\Theta$$
$$\Longrightarrow \sigma\Theta\delta.$$

Therefore, $\beta \subseteq \Theta$ and hence $\Theta = \beta$. Consequently, $S/\Theta = S/\beta$.
Hence, S/Θ is the maximal element of S so that it is a maximal lattice image of S.

5. References

(1) Birkhoff G: Lattice theory, American Mathematical Society Colloquium Publication, third ed. vol. 25, 1973.
(2) Gratzer G: General Lattice Theory, Academic Press, New York, Sanfransisco, 1978.
(3) Halmos PR: Lectures on Boolean algebras, D. Van. Nostrand Company INC, New York, 1963.
(4) Leech J: Skew Lattices in Rings, A. Universalis, vol. 26, 1989, 48-72.
(5) Rao GC: Almost Distributive Lattices, Doctoral Thesis, Dept. of Mathematics, Andhra University, Visakhapatnam, 1980.
(6) Stanely Burris, and Sankappanavar HP: A course in universal algebra, Springer -Verlag, New York, 1980.
(7) Swamy, U. M. and Rao, G. C., Almost Distributive Lattices,J. Aust. Math. Soc.(Series A), vol. 31, 1981, 77-91.